

Available at: [http://www.ictp.trieste.it/~pub\\_off](http://www.ictp.trieste.it/~pub_off)

IC/2001/46

United Nations Educational Scientific and Cultural Organization  
and  
International Atomic Energy Agency  
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

## ON THE STRUCTURE OF BOL ALGEBRAS

Thomas B. Bouetou<sup>1</sup>  
*École Nationale Supérieure Polytechnique,  
B.P. 8390 Yaoundé, Cameroun*  
and  
*The Abdus Salam International Centre for Theoretical Physics,  
Trieste, Italy.*

### Abstract

The fundamental ideas of the definition of solvable and semisimple Bol algebras are given and some related theorems.

MIRAMARE – TRIESTE

May 2001

---

<sup>1</sup>Regular Associate of the Abdus Salam ICTP. E-mail: [tbouetou@polytech.uninet.cm](mailto:tbouetou@polytech.uninet.cm)

# 1 Introduction

It is well known that structure theory is a basic tool for the manipulation of any algebraic-geometrical object, hence this study. In this paper we intend to introduce the notion of solvability for any ideal [2] of Bol algebra and semisimplicity. Our approach uses the notion of solvability introduced in [7] and the one of [8]. By using properties of the defined expression, we will be able to define the notion of the radical of a finite dimensional Bol algebra. Using the notion of an enveloping pair [3, 9] on which Bol algebras are rooted we establish the fact that if any Bol algebra is solvable then its enveloping Lie algebra is solvable; likewise if it is simple. As in [1], it is shown that the Levi-Malcev theorem is not valid for Bol algebra in the classical sense but for a class of Bol algebras called homogeneous. Thus, Bol algebra allows deformation, but the deformation is quite different from the deformation associated with Lie algebras. Bol algebras are also significant in Mathematical Physics.

# 2 Basic definition

**Definition 1.** Any vector space  $V$  of  $\mathbf{B}$  with the operations

$$\xi, \eta \rightarrow \xi \cdot \eta \in V,$$

$$\xi, \eta, \zeta \rightarrow (\xi, \eta, \zeta) \in V \quad (\xi, \eta, \zeta \in V)$$

and identities

$$\xi \cdot \xi = 0, \quad (\xi, \eta, \zeta) + (\eta, \zeta, \xi) + (\zeta, \xi, \eta) = 0,$$

$$(\xi, \eta, \zeta)\chi - (\xi, \eta, \chi)\zeta + (\zeta, \chi, \xi \cdot \eta) - (\xi, \eta, \zeta \cdot \chi) + \xi\eta \cdot \zeta\chi = 0,$$

$$(\xi, \eta, (\zeta, \chi, \omega)) = ((\xi, \eta, \zeta), \chi, \omega) + (\zeta, (\xi, \eta, \chi), \omega) + (\zeta, \chi, (\xi, \eta, \omega))$$

is called a Bol algebra.

We will note that any Bol algebra can be realized as the tangent algebra to a Bol Loop with the left Bol identity, and they allow imbedding in Lie algebras. This can be expressed as follows:

let  $(G, \Delta, e)$ - be a local Lie group,  $H$ - one of its subgroups, and let us denote the corresponding Lie algebra and subalgebra by  $\mathfrak{G}$  and  $\mathfrak{h}$ . Consider a vector subspace  $\mathfrak{B}$  such that

$\mathfrak{G} = \mathfrak{h} \dot{+} \mathfrak{B}$ . Let  $\Pi : G \longrightarrow G \setminus H$  be the canonical projection and let  $\Psi$  be the restriction of mappings composition  $\Pi \circ \exp$ , to  $\mathfrak{B}$ . Then there exists such a neighbourhood  $\mathcal{U}$  of the point  $O$  in  $\mathfrak{B}$  such that  $\Psi$  maps it diffeomorphically into the neighbourhood  $\Psi(u)$  of the coset  $\Pi(e)$  in  $G \setminus H$ .

$$\begin{array}{ccc} \mathfrak{G} & \xleftarrow{i} & \mathfrak{B} \\ \exp \downarrow & & \downarrow \Psi \\ G & \xrightarrow{\pi} & G \setminus H \end{array}$$

by introducing a local composition law:

$$a \star b = \Pi_B(a \Delta b)$$

On points of local cross-section  $B = \exp \mathcal{U}$  of left cosets of  $G \bmod H$  where  $\Pi_B = \exp \circ \Psi^{-1} \circ \Pi : G \longrightarrow B$  is the local projection on  $B$  parallel to  $H$  which puts every element  $a \in B$  in correspondence so that  $g = a \Delta p$ , where  $p \in H$ . We will emphasize that if any two local analytic Loops are isomorphic then their corresponding Bol algebras are isomorphic [4, 5, 6, 9, 10, 11, 12, 13].

### 3 Solvable Bol algebras

Let  $\mathfrak{B}$  be a Bol algebra over a field  $K$  of characteristic 0.

**Definition 2.** We say that the subspace  $\mathfrak{V}$  is a subsystem of  $\mathfrak{B}$  if  $(\mathfrak{V}, \mathfrak{V}, \mathfrak{V}) \subset \mathfrak{V}$  and an ideal if  $(\mathfrak{V}, \mathfrak{B}, \mathfrak{B}) \subset \mathfrak{V}$  and  $\mathfrak{V} \cdot \mathfrak{B} \subset \mathfrak{V}$ .

Our idea is to follow the general notion and keep out the condition of Bol algebras to give the definition of solvability of Bol algebras. Following [8] we define the series:

$$\mathfrak{V}^{(n+1)} = (\mathfrak{V}^{(n)}, \mathfrak{V}^{(n)}, \mathfrak{B}) \quad \forall n \geq 0$$

where  $\mathfrak{V}$  is an ideal and one can verify that:

$$\mathfrak{V}^{(n+1)} \subseteq \mathfrak{V}^{(n)} \subseteq \mathfrak{V}^{(n-1)} \subseteq \dots \subseteq \mathfrak{V}^{(0)} = \mathfrak{V}$$

**Definition 3.** We say that the ideal  $\mathfrak{V}$  is solvable if there exist  $k \geq 0$  such that  $\mathfrak{V}^{(k)} = 0$ .

The given definition was developed in [8] for a trilinear operation. For Bol algebras we have a bilinear and a trilinear operation. Therefore, we will now define the ideal of Bol algebras, by considering the two operations.

Let  $\mathfrak{W}$  be an Ideal related to the l.t.s of Bol algebras  $\mathfrak{B}$  and let us define the subsystem  $\mathfrak{M}$  by  $\mathfrak{M} = \mathfrak{W} \cdot \mathfrak{W} + (\mathfrak{W}, \mathfrak{W}, \mathfrak{B})$ .

It is clear that  $\mathfrak{M}$  is an ideal of  $\mathfrak{W}$  even normal. We define the relation:

$$\mathfrak{W} = \mathfrak{W}^{(0)}$$

$$\mathfrak{W}^{(n+1)} = \mathfrak{W}^{(n)} \cdot \mathfrak{W}^{(n)} + (\mathfrak{W}^{(n)}, \mathfrak{W}^{(n)}, \mathfrak{B})$$

**Theorem 1.**  $\forall n > 0$   $\mathfrak{W}^{(n)}$  is an ideal in  $\mathfrak{W}^{(n-1)}$ .

Proof:

For proof of this theorem, we will use the induction on  $n$ .

For  $n = 1$  we must verify that  $\mathfrak{W} \cdot \mathfrak{W}' \subset \mathfrak{W}'$  and  $(\mathfrak{W}', \mathfrak{W}, \mathfrak{W}) \subset \mathfrak{W}'$  where  $\mathfrak{W}' = \mathfrak{W} \cdot \mathfrak{W} + (\mathfrak{W}, \mathfrak{W}, \mathfrak{B})$ .

We have,  $\mathfrak{W} \cdot \mathfrak{W}' = \mathfrak{W} \cdot \mathfrak{W} \cdot \mathfrak{W} + \mathfrak{W}(\mathfrak{W}, \mathfrak{W}, \mathfrak{B})$  considering the identities of the definition of Bol algebras we have  $\mathfrak{W} \cdot \mathfrak{W}' \subset \mathfrak{W}'$  and

$$(\mathfrak{W}', \mathfrak{W}, \mathfrak{W}) = (\mathfrak{W}, \mathfrak{W}, (\mathfrak{B}, \mathfrak{W}, \mathfrak{W})) \subset \mathfrak{W}'.$$

We assume that  $\mathfrak{W}^{(n)} \subseteq \mathfrak{W}^{(n-1)}$  and show that  $\mathfrak{W}^{(n+1)} \subseteq \mathfrak{W}^{(n)}$  or

$$\mathfrak{W}^{(n+1)} \subseteq \mathfrak{W}^{(n-1)} \cdot \mathfrak{W}^{(n-1)} + (\mathfrak{W}^{(n-1)}, \mathfrak{W}^{(n-1)}, \mathfrak{B}) \equiv \mathfrak{W}^{(n)}.$$

We have

$$\mathfrak{W}^{(n-1)} \cdot \mathfrak{W}^{(n)} \subseteq \mathfrak{W}^{(n-1)} \cdot \mathfrak{W}^{(n-1)} \subseteq \mathfrak{W}^{(n)}.$$

Also since  $\mathfrak{W}$  is a subset we have:

$$(\mathfrak{W}^{(n-1)}, \mathfrak{W}^{(n-1)}, \mathfrak{W}^{(n)}) \subseteq (\mathfrak{W}^{(n-1)}, \mathfrak{W}^{(n-1)}, \mathfrak{B}) \subseteq \mathfrak{W}^{(n)}$$

hence the result.

As in the case of Lie algebras, we can then write the following ascending series:

$$\mathfrak{W}^{(n+1)} \subset \mathfrak{W}^{(n)} \subset \mathfrak{W}^{(n-1)} \subset \dots \subset \mathfrak{W}^{(1)} \subset \mathfrak{W}^{(0)} = \mathfrak{W}.$$

After conducting the examination one can state the following definition:

**Definition 4.** The subsystem  $\mathfrak{W}$  of a Bol algebra  $\mathfrak{B}$  is an ideal of  $\mathfrak{B}$  if

$$\mathfrak{W} \cdot \mathfrak{W} + (\mathfrak{W}, \mathfrak{W}, \mathfrak{B}) \subset \mathfrak{W}.$$

**Definition 5.** The ideal  $\mathfrak{W}$  of a Bol algebra  $\mathfrak{B}$  is said to be solvable if there exist  $k \geq 0$  such that  $\mathfrak{W}^{(k)} = 0$ .

We will discuss the notion of maximal ideal:

**Lemma 1.** If  $\mathfrak{V}$  and  $\mathfrak{W}$  are solvable ideals then so is  $\mathfrak{V} + \mathfrak{W}$ .

Proof:

Assume that  $n, k$  exist, such that  $V^{(n)} = 0$  and  $W^{(k)} = 0$ . We also suppose that  $n = k$

$$V^{(n)} = V^{(n-1)} \cdot V^{(n-1)} + (V^{(n-1)}, V^{(n-1)}, \mathfrak{B})$$

$$W^{(n)} = W^{(n-1)} \cdot W^{(n-1)} + (V^{(n-1)}, W^{(n-1)}, \mathfrak{B})$$

We show that  $(V + W)^{(n)} = 0$  by using induction on  $n$ . For  $n = 0$  it's obvious. For  $n = 1$  we have

$$\begin{aligned} (V + W)^{(1)} &= (V + W) \cdot (V + W) + (V + W, V + W, \mathfrak{B}) \\ &= V \cdot V + W \cdot W + V \cdot W + W \cdot V + (V, V, \mathfrak{B}) + (V, W, \mathfrak{B}) + (W, V, \mathfrak{B}) + (W, W, \mathfrak{B}) \\ &= V^{(1)} + W^{(1)} + V \cdot W + W \cdot V + (V, W, \mathfrak{B}) + (W, V, \mathfrak{B}) \\ &\subseteq V^{(1)} + W^{(1)} + V \cap W. \end{aligned}$$

We assume that:

$$(V + W)^{(k)} \subseteq V^{(k)} + W^{(k)} + V \cap W$$

and prove that:

$$(V + W)^{(k+1)} \subseteq V^{(k+1)} + W^{(k+1)} + V \cap W.$$

We have

$$\begin{aligned} (V + W)^{(k+1)} &= (V + W)^{(k)} \cdot (V + W)^{(k)} + ((V + W)^{(k)}, (V + W)^{(k)}, \mathfrak{B}) \\ &\subseteq (V^{(k)} + W^{(k)} + V \cap W)(V + W)^{(k+1)} + (V^{(k)} + W^{(k)} + V \cap W, (V + W)^{(k)}, \mathfrak{B}). \end{aligned}$$

After opening and rearranging we obtain the result.

This leads us to define the radical of Bol algebras.

**Definition 6.** The radical  $\mathfrak{R}$  of a Bol algebra  $\mathfrak{B}$  is the maximal and unique solvable ideal.

**Definition 7.** A Bol algebra is said to be semisimple if its radical is zero.

As they are constructed, Bol algebra can be enveloped in Lie algebra. We now discuss some facts about it using the notion of pseudo-derivation. Following [11] we know that Bol algebras are defined from two operations  $(\cdot)$  and  $(; ,)$  verifying a series of conditions. In what follows, we will discuss some results from [11].

**Definition 8.** The linear endomorphism  $\Pi$  of the binary-ternary algebra  $\mathfrak{B}$  with the composition law  $X \cdot Y$  and  $(X; Y, W)$  will be called pseudo-derivation with the component  $Z$  if

$$\begin{aligned} \Pi(X \cdot Y) &= (\Pi X) \cdot Y + X \cdot (\Pi Y) + (Z; X, Y) + (X \cdot Y) \cdot Z \\ \Pi(X; Y, W) &= (\Pi X; Y, W) + (X; \Pi Y, W) + (X; Y, \Pi Z) \\ &\quad \forall X, Y, Z, W \in \mathfrak{B} \end{aligned}$$

The pseudo-derivation of Bol algebra  $\mathfrak{B}$  form a Lie algebra with respect to the operation of sum and multiplication by a scalar and the Lie commutator  $[\Pi, \tilde{\Pi}] = \Pi \tilde{\Pi} - \tilde{\Pi} \Pi$ .

For Bol algebra, one can verify that if  $\Pi$ , and  $\tilde{\Pi}$  has  $Z$  and  $\tilde{Z}$  as a component respectively then  $\Pi + \tilde{\Pi}$  and  $\lambda \Pi$  have  $Z + \tilde{Z}$  and  $\lambda Z$  correspondingly as a component but  $[\Pi, \tilde{\Pi}]$  has for a component  $Z \cdot \tilde{Z} + \Pi \tilde{Z} - \tilde{\Pi} Z$ . We denote the algebra formed by the pseudo-derivation by  $pder\mathfrak{B}$ . Using this notation we re-write the definition of Bol algebra as follows:

**Definition 9.** The binary- ternary  $\mathbb{R}$ -algebra  $\mathfrak{B}$  with the bilinear  $(\cdot)$  and the trilinear  $(; ,)$  operation is called a Bol algebra if:

- $X \cdot X = 0$
- $(X; Y, Y) = 0$
- $(X; Y, Z) + (Z; X, Y) + (Y; Z, Y) = 0$
- the endomorphism  $\mathbb{D}_{X,Y} : Z \longrightarrow (Z; X, Y)$  is a pseudo-derivation with the component  $X \cdot Y$ .

From [11] it is shown that the set

$$Pder\mathfrak{B} = \{(\Pi, a); \Pi \in pder\mathfrak{B}, a \text{ a component of } \Pi\}$$

is a Lie algebra under a proper operation.

This notion of pseudo-derivation helps to define the enveloping Lie algebra for a Bol algebra. In [11] it is proved that for any Bol algebra  $\mathfrak{B}$  with the operation  $X \cdot Y$  and  $(Z; X, Y)$  there is a pair of algebras  $(\mathfrak{G}, \mathfrak{h})$  such that  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{h}$ , and  $[\mathfrak{B}, [\mathfrak{B}, \mathfrak{B}]] \subset \mathfrak{B}$ ,  $[\mathfrak{B}, \mathfrak{B}] \cap \mathfrak{B} = \{0\}$  with  $pro.\mathfrak{B}_{[X,Y]} = X \cdot Y$  and  $(Z; X, Y) = [Z, [X, Y]]$  where  $X, Y, Z \in \mathfrak{B}$ . This implies that  $[X, Y] = X \cdot Y + [-X \cdot Y + (\mathfrak{D}_{X,Y}, X \cdot Y)]$ .

If  $V$  is an ideal of Bol algebra  $\mathfrak{B}$  and  $\mathfrak{G}$  its universal enveloping Lie algebra, then any ideal  $W$  generated by  $V$  coincides with  $V \dot{+} [V, \mathfrak{B}]$ .

We have the following theorem:

**Theorem 2.** Let  $V$  be a solvable ideal in  $\mathfrak{B}$  then  $W = V \dot{+} [V, \mathfrak{B}]$  is also a solvable ideal in  $\mathfrak{B}$ .

Before proving this theorem, we state the following lemma.

**Lemma 2.** If  $V$  is a solvable ideal in  $\mathfrak{B}$  and  $\tilde{V}$  an extension of  $V$  such that  $\tilde{V} = V + [V, \mathfrak{B}]$ , then  $\tilde{V}$  is solvable.

Proof:

We want to show that there  $\exists k$  such that  $(\tilde{V})^{(k)} = 0$ .

We have

$$(V + [V, \mathfrak{B}])^{(k)} = (V + [V, \mathfrak{B}])^{(k-1)} \cdot (V + [V, \mathfrak{B}])^{(k-1)} + ((V + [V, \mathfrak{B}])^{(k)}, (V + [V, \mathfrak{B}])^{(k)}, \mathfrak{B})$$

Here we see that the power on the left side is a function of  $k$ . We will then write  $p = p(k)$  therefore, we have the following lemma:

**Lemma 3.** The following inclusion holds:  $\tilde{V}^{(p)} \subset V^{(k)} + [V^{(k)}, \mathfrak{B}]$  where  $\tilde{V}^{(0)} = \tilde{V}$  and  $\tilde{V}^{(n+1)} = [\tilde{V}^{(n)}, \tilde{V}^{(n)}]$ .

Proof:

The proof of this lemma is by induction. Here we will assume that, if  $k = 0$  then  $p = 0$ . For  $k = 0$  it's obvious. We assume that for any  $p$  and  $k$

$$\tilde{V}^{(p)} \subset V^{(k)} + [V^{(k)}, \mathfrak{B}] \quad *$$

and prove that:

$$\tilde{V}^{(p')} \subset V^{(k+1)} + [V^{(k+1)}, \mathfrak{B}].$$

We have  $\forall k, [V^{(k)}, V^{(k)}] \subset [V^{(0)}, V^{(k)}]$  and  $[V^{(k)}, [V^{(k)}, \mathfrak{B}]] \subset (V^{(k)}, V^{(k)}, \mathfrak{B})$  since by definition  $V^{(k+1)} = (V^{(k)}, V^{(k)}, \mathfrak{B})$ . So by taking the commutator of the relation (\*) we have

$$\begin{aligned} [\tilde{V}^{(p)}, V^{(k)}] &\subset [V^{(k)}, V^{(k)}] + [[\mathfrak{B}, V^{(k)}], V^{(k)}] \\ &\subset V^{(k+1)} + [[\mathfrak{B}, V^{(k+1)}]. \end{aligned}$$

Hence the result.

Proof of the theorem.

Since the power of the right side of the theorem is a function of  $k$ , then by applying the lemma up to the reduction, we obtain the result. Hence the theorem is proved.

From this theorem we have the following corollary:

**Corollary** If  $\mathfrak{B}$  is a solvable Bol algebra then the universal enveloping Lie algebra  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{B} \wedge \mathfrak{B}$  is solvable.

## 4 The Killing-Ricci form for Bol algebras and semisimple Bol algebras

In this section we shall follow the construction in [7]. A Lie triple algebra (L.t.a) is defined, as an anticommutative algebra over a field  $F$ , whose multiplication is denoted by  $XY$  for  $X, Y \in \mathfrak{U}$ . Denote by  $\mathbb{D}(X, Y)Z$  the inner derivative satisfying a series of identities [14, 15]. Bol algebras have the same construction as L.t.a; only that the endomorphism  $\mathbb{D}(X, Y)Z$  is called the inner pseudodifferential of the algebra  $\mathfrak{B}$ . The universal enveloping Lie algebra of  $\mathfrak{B}$  is a Lie algebra  $\mathfrak{G} = \mathfrak{B} \dot{+} \mathbb{D}(\mathfrak{B}, \mathfrak{B})$  such that the commutator in  $\mathfrak{G}$  is defined as:  $[\xi, \eta] = \xi \cdot \eta + (\xi, \eta)$ . Conversely, if we have any Lie algebra  $\mathfrak{G}$  and a subalgebra  $\mathfrak{h}$ , we can define the operation in  $\mathfrak{B}$  such that  $\xi \cdot \eta = \Pi_{\mathfrak{B}}[\xi, \eta]$ . The projection, is parallel to  $\mathfrak{h}$ . As we have seen, Bol algebra allows imbedding in Lie algebra, let  $\mathfrak{B}$  be a Bol algebra and let  $e_i, i = \overline{1, n}$  be its  $n$ -dimensional basis and  $\mathbb{D}_1, \dots, \mathbb{D}_N$  the basis of the pseudo-derivation space  $\mathbb{D}(\mathfrak{B}, \mathfrak{B})$ . We assume that  $\mathbb{D}(\mathfrak{B}, \mathfrak{B}) \neq 0$  we define the operation in Bol algebra by means of a tensor as:

$$e_i \cdot e_j = T_{i,j}^k e_k; \mathbb{D}(e_i, e_j)e_k = R_{i,j,k}^l e_l$$

$$\mathbb{D}(e_i, e_j) = \mathbb{D}_{i,j}^\tau \mathbb{D}_\tau$$

$$[\mathbb{D}_\tau, e_i] = \mathbb{D}_\tau e_i = K_{\tau,i}^j e_j$$

where the indices  $i, j, k, l \in \{1, \dots, n\}$  and  $\tau \in \{1, \dots, N\}$  and

$$\mathfrak{G} = \mathfrak{B} \dot{+} \mathfrak{B} \wedge \mathfrak{B}$$

is universal enveloping Bol algebra. We denote by  $\alpha$ , the Killing form of the universal enveloping Lie algebra. By the killing-Ricci form  $\beta$  of the Bol algebra we understand a symmetric bilinear form on  $\mathfrak{B}$  determined by the restriction of  $\alpha$  to  $\mathfrak{B} \wedge \mathfrak{B}$ .

**Proposition 1.** For any  $\xi, \eta \in \mathfrak{B}$ , we have the following relation:

$$\beta(\xi, \eta) = \text{tr}(L(\xi, \eta) + L(\eta, \xi))$$

where  $L(\xi, \eta)\zeta = (\xi, \eta, \zeta)$ .

**Proof:**

Let  $e_1, \dots, e_n$  be a basis on  $\mathfrak{B}$  and  $\mathbb{D}_1, \dots, \mathbb{D}_N$  be a basis in the pseudo-derivation algebra. We now calculate the basis element of the form  $\beta$ .

$$\beta(e_i, e_j) = \text{tr}(L_i, L_j) = \text{tr}(L_i L_j).$$

We also have

$$L_i L_j e_k = [e_i, [e_j, e_k]] = R_{i,j,k}^l e_l$$

and

$$L_i L_j \mathbb{D}_\tau = [e_i, [e_j, \mathbb{D}_\tau]] =$$

$$[e_i, K_{j\tau}^m e_m = K_{j\tau}^m T_{im}^\nu e_\nu.$$

Therefore  $\beta(e_i, e_j) = R_{ijk}^l + K_{j\tau}^m T_{im}^\tau$ .

On the other hand, the expressions

$$[e_j, [e_i, e_m]] = T_{im}^\rho K_{j\rho}^s e_s = R_{jim}^s e_s$$

imply:

$$R_{jim}^s = T_{im}^\rho K_{j\rho}^s, R_{ji}^m = T_{im}^\rho K_{j\rho}^m.$$

Therefore the basic element of the form  $\beta$  can be written as:

$$\beta(e_i, e_j) = R_{ijk}^k + R_{jik}^k.$$

Considering the Ricci tensor by convolution of the upper and the lower index we then obtain:

$$R_{ij} = R_{ijk}^k = \text{tr} L(e_i, e_j)$$

$$\beta(e_i, e_j) = R_{ij} + R_{ji} = \text{tr}(L(e_i, e_j) + L(e_j, e_i))$$

Hence the proposition is proved.

An elementary check- up shows that  $\beta$  induces the properties of the form  $\alpha$  for the Lie algebra  $\mathfrak{G}$ .

**Definition 10.** We say that a Bol algebra  $\mathfrak{B}$  is semisimple if it contains only trivial ideals.

Let  $\mathfrak{b}$  be an invariant symmetric bilinear form on  $\mathfrak{B}$  satisfying the following condition:

$$\mathfrak{b}(X \cdot Y, Z) = \mathfrak{b}(X, Y \cdot Z) \quad (**)$$

$$\mathfrak{b}((X, Y, Z), t) = \mathfrak{b}(Z, (X, Y, t))$$

Following [7] we will introduce the notion of perpendicularity by means of the form of an object relatively to  $\mathfrak{B}$ .

Let  $I$  be an ideal of the Bol algebra  $\mathfrak{B}$ , in particular we can assume  $I \triangle \mathfrak{B}$ . We will define the following two sets i.e. the left orthogonal, and the right orthogonal respectively, by:

$$I_l^\perp = \{X \in \mathfrak{B} / \mathfrak{b}(X, I) = 0\}$$



$$I_r^\perp = \{X \in \mathfrak{B} / \mathfrak{b}(I, X) = 0\}$$

After introducing these concepts, this leads us to the following definition:

**Definition 11.** The set  $\mathfrak{C}$  defined as

$$\mathfrak{C} = \{X \in \mathfrak{B} / \mathfrak{B} \cdot X = 0, (\mathfrak{B}, \mathfrak{B}, X) = 0\}$$

it is called the center.

**Proposition 2.**  $\mathfrak{C}_r^\top = \mathfrak{C}_l^\top = \mathfrak{B}^{(1)}$  where  $\mathfrak{B}^{(1)} = \mathfrak{B} \cdot \mathfrak{B}$ .

Proof:

Let  $X \in (\mathfrak{B}^{(1)})_l^\top$ . This implies that  $\mathfrak{b}(X, \mathfrak{B}, \mathfrak{B}) = 0$ . By using  $(**)$  we have  $\mathfrak{b}(X, (\mathfrak{B}, \mathfrak{B}, \mathfrak{B})) = 0$  and hence  $(X \cdot \mathfrak{B}, \mathfrak{B}) = 0$ . Also  $\mathfrak{b}((\mathfrak{B}, \mathfrak{B}, X), \mathfrak{B}) = 0$  gives  $X \cdot \mathfrak{B} = 0$  and  $(\mathfrak{B}, \mathfrak{B}, X) = 0$  hence,  $X \in C$ . Therefore,  $X \in (\mathfrak{B}^{(1)})_l^\top \subseteq C$ . In the same way, we prove the converse and therefore  $X \in (\mathfrak{B}^{(1)})_l^\top = C$ . Analogously; one can prove that  $X \in (\mathfrak{B}^{(1)})_r^\top = C$ . Hence by taking the orthogonality twice we obtain the result

$$\mathfrak{C}_r^\top = \mathfrak{C}_l^\top = \mathfrak{B}^{(1)}$$

Following Lie algebra theory and the result in [7] we can state the following Theorem:

**Theorem 3.** Let  $\mathfrak{B}$  be a finite dimensional Bol algebra and let  $\mathfrak{b}$  be an invariant nondegenerated form on  $\mathfrak{B}$  that does not contain trivial ideals of the form:

$$I^{(1)} = I \cdot I + (I, I, \mathfrak{B}) = 0$$

then,  $\mathfrak{B}$  can be split in a direct sum of ideal

$$\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2 + \dots + \mathfrak{B}_r$$

where  $b(\mathfrak{B}_i, \mathfrak{B}_j) = 0$  for  $i \neq j$  and each of the ideal  $\mathfrak{B}_i$  is a simple Bol algebra.

Proof:

Let  $I$  be an ideal of  $\mathfrak{B}$ . We can define the sets  $I_r^\top$  and  $I \cap I_r^\top$  are also ideals of  $\mathfrak{B}$ . Then denoting by  $M = I \cap I_r^\top$  which is normal in  $\mathfrak{B}$  and we will have  $b(M, M) = 0, b(\mathfrak{B}, M \cdot M) = 0; b(\mathfrak{B}, (M, M, \mathfrak{B})) = b(\mathfrak{B}, (M, \mathfrak{B}, M)) = 0$ .

Hence  $M^{(1)} = 0$  and  $M = 0$ . We then have  $\mathfrak{B} = I \dot{+} I_r^\top$ , and  $b(I, I_r^\top) = 0$ . Now we must prove  $b(I_r^\perp, I) = 0$ . We have  $I \subset \mathfrak{B}$  which is an ideal and so we can define  $I^{(1)} = I \cdot I + (I, I, I + I_r^\top) \neq 0$  which is also an ideal of  $I$  therefore  $I = I^{(1)}$  and  $I$  is a semisimple Bol algebra and  $b(I, I_r^\perp) = 0$ . Since  $I \cdot I_r^\top$  and  $(I, I, I_r^\top) \subseteq I \cap I_r^\top = 0$  then  $b(I_r^\top, I \cdot I) = b(I \cdot I_r^\top, I) = 0$  and  $b(I_r^\top, (I, I, I)) = b((I, I, I_r^\top), I) = 0$  also  $b(I_r^\top, I^{(1)}) = b(I_r^\top, I) = 0$  therefore  $I_r^\top \subset I_l^\top$  now the rest of the theorem is established by the induction on  $\dim \mathfrak{B}$ .

As in [7] we can state the following theorem for Bol algebra.

**Theorem 4.** Let  $\mathfrak{B}$  be a finite dimensional Bol algebra over a field of characteristic zero and let  $\beta$  be the Killing-Ricci form then:

1. The universal enveloping Lie algebra  $\mathfrak{G}$  of  $\mathfrak{B}$  is solvable if and only if  $\mathfrak{B}$  and  $(\mathfrak{B}, \mathfrak{B}, \mathfrak{B})$  are orthogonal.

2.  $\mathfrak{G}$  is simple if and only if  $\beta$  is nondegenerated.
3. If the form  $\beta$  is non degenerate and invariant, then  $\mathfrak{B}$  can be split into the direct sum of simple ideals which are orthogonal in the sense of the definition above.

$$\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2 + \dots + \mathfrak{B}_r$$

and, the universal enveloping Lie algebra also split into a direct sum of ideal  $\mathfrak{G}_i$  such that each  $\mathfrak{G}_i$  is a universal enveloping simple Lie algebra and we have  $\mathfrak{B} = (\mathfrak{B}, \mathfrak{B}, \mathfrak{B})$ .

Proof:

In the proof of this theorem, we have to consider all the results of Lie algebras relative to the Killing form.

1. If  $\mathfrak{G}$  is a solvable Lie algebra then  $\alpha(\mathfrak{G}, [\mathfrak{G}, \mathfrak{G}]) = 0$ . As we know,  $[\mathfrak{G}, \mathfrak{G}] = (\mathfrak{B}, \mathfrak{B}, \mathfrak{B}) + [\mathfrak{B}, \mathfrak{B}]$  and so  $\beta(\mathfrak{B}, (\mathfrak{B}, \mathfrak{B}, \mathfrak{B})) = 0$ .

conversely, if  $\beta(\mathfrak{B}, (\mathfrak{B}, \mathfrak{B}, \mathfrak{B})) = 0$  then  $[\mathfrak{B}, \mathfrak{B}] = \text{Der}\mathfrak{B}$  (set of pseudo-derivation) and so  $\alpha(\mathfrak{B}, [\mathfrak{G}, \mathfrak{G}]) = 0$ . It follows that  $\mathfrak{B}$  lies in the radical of  $\mathfrak{G}$  and hence  $\mathfrak{B}$  is solvable. Moreover, using the result of the corollary above about solvability we say the enveloping Lie algebra is also solvable.

2. Let  $\mathfrak{G}$  be a simple Lie algebra then the Killing form on  $\mathfrak{G}$  is non degenerate Since  $\mathfrak{G}$  can be split into a direct sum of orthogonal subspace  $\mathfrak{B}$  and  $[\mathfrak{B}, \mathfrak{B}]$  relative to  $\alpha$ , then these sudspaces are also non degenerate. In particular, the form  $\beta$  is non degenerate. Conversely, if the form  $\beta$  is non degenerate, then we have a natural embedding  $\sigma$  see [8] on  $\mathfrak{G}$ :

$$\mathfrak{G}^\sigma = \mathfrak{B}^\sigma + [\mathfrak{B}^\sigma, \mathfrak{B}^\sigma]$$

such that  $X \longrightarrow X^\sigma$ .

The operation in  $\mathfrak{G}^\sigma$  using the properties of the pseudo-derivative in  $\mathfrak{B}$  is then defined as:

$$\begin{aligned} [X^\sigma, X^\sigma] &= D(X, Y) \\ [X^\sigma, D(Y, Z)] &= (X, Y, Z)^\sigma \\ [D(X, Y), D(U, V)] &= D(X, Y)D(U, V) - D(U, V)D(X, Y) \\ &= D((X, U, V), Y) + D(X, (Y, U, V)) \end{aligned}$$

where  $X, Y, Z, U, V \in \mathfrak{B}$ .

Let  $\alpha'$  be the Killing form in  $\mathfrak{G}^\sigma$ ,  $\beta'$  the restriction of  $\alpha'$  in  $\mathfrak{B}^\sigma$  then

$$\beta'(X^\sigma, X^\sigma) = \beta(X, Y)$$

since  $\beta'$  is non degenerate on  $\mathfrak{B}^\sigma$ , one can easily verify that  $\alpha'$  is also non degenerate; it is clear that the subspace  $\mathfrak{B}^\sigma$  and  $[\mathfrak{B}^\sigma, \mathfrak{B}^\sigma]$  are orthogonal relatively to  $\alpha'$ . So by a simple verification one can see that  $[\mathfrak{B}^\sigma, \mathfrak{B}^\sigma]$  is also non degenerate. Indeed if  $\mathfrak{D} \in [\mathfrak{B}^\sigma, \mathfrak{B}^\sigma]$  then  $\alpha'([\mathfrak{B}^\sigma, \mathfrak{B}^\sigma], \mathfrak{D}) = 0$ . On the other hand,

$$\alpha'([X^\sigma, Y^\sigma], \mathfrak{D}) = \beta'((X^\sigma, (Y\mathfrak{D})^\sigma))$$

$\forall X, Y \in \mathfrak{B}$  then  $(Y\mathfrak{D})^\sigma = 0$ . That means  $Y\mathfrak{D} = 0$  and hence  $\mathfrak{D} = 0$ . Therefore is  $\alpha'$  non degenerated and  $\mathfrak{G}^\sigma$  is semisimple hence the trilinear operation is semisimple.

3. Let us assume that  $\beta$  is non degenerate and invariant as in the second point; we have the trilinear operation semisimple, and so  $\mathfrak{B}$  is semisimple. By using Theorem 3. which states that  $\mathfrak{B}$  can be split into a direct sum of simple orthogonal ideal  $\mathfrak{B}_i$  where  $i = \overline{1, n}$ . Indeed if  $X_i \in \mathfrak{B}_i$  and  $X_j \in \mathfrak{B}_j$  and  $i \neq j$  then  $X_i \cdot Y_j = 0$  and  $(t, X_i, Y_j) \in \mathfrak{B}_i \cap \mathfrak{B}_j = 0$   $\forall t \in \mathfrak{B}$  therefore  $\mathfrak{D}(X_i, Y_j)$  is reduced to the null vector which is the subset of  $\mathfrak{B}$  With zero component  $X_i Y_j$  hence we have

$$\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2 + \mathfrak{G}_3 + \dots + \mathfrak{G}_n$$

where  $\mathfrak{G}_i$  is the enveloping algebra of  $\mathfrak{B}_i$  hence the theorem is proved.

**Acknowledgments:** This paper was completed, thanks to the help of Professor M. Kikkawa who sent all the documentation I needed. This work was done within the framework of the Associateship Scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

## References

- [1] Bouetou B.T. A note on Levi-Malcev theorem for homogeneous Bol algebras// submitted for publication.
- [2] Chevalley C. Theory of Lie groups// Princeton Uni. press 1946, 318p.
- [3] Dixmier J. Enveloping Algebras// Graduate studies in Maths. vol. 11, AMS 1996.
- [4] Harris B. Cohomology of Lie triple system and Lie algebras with involution// trans. AMS 1961 vol.71 pp. 148-162.
- [5] Kikkawa M. Canonical connections of homogeneous Lie loops and 3-Webs.// Hiroshima math. j.-1986.-vol.2.-pp.37-55.
- [6] ——— note on subloops of a homogeneous Lie loop and subsystems of its Lie triple algebras.// Hiroshima math. j.-1975 vol.5.-pp.433-446.

- [7] ——— On Killing-Ricci forms of Lie triple algebras// *pac.j. of math.* vol. 96, N0. 1, 1981 pp. 153-161.
- [8] Lister W.G. A structure theory of Lie triple systems// *trans. AMS.*-1952.-t.72.-pp.217-245.
- [9] Sabinin L.V. Mikheev P.O. The theory of smooth Bol loops// *Lecture notes Friendship of Nations University press, Moscow* 1985, 81p.
- [10] ——— Quasigroups and differential geometry// In *Quasigroups and Loops Theory and Applications*, collective monograph (O.Chein, H.Pflugfelder and J.D.H.Smith) Heldermann,verlag, Berlin 1990 CH. XII pp.357-430.
- [11] Sabinin L.V. Smooth Quasigroups and Loops// *Monograph mathematics and it's application* vol. 492 XVI+250, Kluwer academic publishers, Dordrecht/Boston/London, 1999.
- [12] ——— Smooth Quasigroups and Loops forty-five years of incredible growth// *proceeding loop's99 Prague*.
- [13] Shafer R.D. An Introduction to nonassociative algebras// *academic press* 1966 Ney York and London.
- [14] Yamaguti K. On algebras of totally geodesic space// *J. Sci. Hirohsima Uni* (1957). ser. A. 21, N.2 pp. 107-113.
- [15] Yamaguti K. On the Lie triple system and its generalization// *J. Sci. Hirohsima Uni* (1958). ser. A. 21, N.3 pp. 155-160.